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# Unifying time-to-build theory\*

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## Abstract

Discrete-time and continuous-time models with time-to-build do not possess the same dimension. In this paper we address the dimensionality issue by revising the way time-to-build is traditionally measured when transforming the economic dynamics from discrete to continuous time. We propose our new procedure as an alternative approach which allows for a unifying theory of the two classes of economic growth models. Interestingly enough, discrete-time models admit a change in stability for a value of the time-to-build parameter where their continuous counterpart admits an Hopf bifurcation.

*Keywords:* Discrete and continuous-time; time-to-build, mixed functional differential equations.

*JEL Classification:* E00, E30, O40.

## 1 Introduction

The dimensionality of a continuous-time neoclassical growth model switches from finite to infinite as soon as capital takes time to become productive. In fact, the presence of the time-to-build parameter,  $J > 0$ , pins down a system of mixed functional differential equations in capital and consumption, whose associated characteristic equation is transcendental in the complex field, and it has an infinite of roots.

On the contrary, in its discrete-time version, the time-to-build assumption simply increases the order of the dynamical system describing the economy by a factor  $2J$ , with  $J$  an integer multiple of the chosen calendar time unit, and thus it remains finite. In particular, the characteristic equation associated to a neoclassical growth model (as in Kydland and Prescott [18] among others) is algebraic of corresponding degree.

Our objective is to propose a procedure which preserves the time-to-build structure of capital in the passage from discrete-time to continuous-time and, in doing so, explains the change in dimensionality. To this purpose, we need to be careful in assigning the time-to-build parameter its appropriate measure. In fact, with regard to the problem of the nature of time in neoclassical growth models, we maintain a sort of agnostic point of view, thus not entering the debate on the “proper” dimension the time variable must be given in any particular economic setting. Our

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focus remains, instead, on the study of the morphological differences that the model's dynamics may exhibit when written by choosing a continuous vs. a discrete time approach. In this respect the transformation issue - namely the problem of deriving the "correct" continuous counterpart of an assigned discrete time dynamics, and vice versa - is particularly relevant to us, and will deserve a specific attention further on.

Since the early contributions on the relation between discrete and continuous-time models (see May [20], Foley [12], and Turnovsky [21], among others), the conventional procedure to move from discrete-time to continuous-time has been to re-scale all the key equations describing the economy along any subinterval  $[t, t+h]$  and let the period length,  $h$ , tend to zero; then the discrete-time capital equation  $K_{t+1} = f(K_t)$  is usually re-written as

$$\Delta_h(K_t) = K_{t+h} - K_t = h\Delta_1(K_{t+1}) = h(f(K_t) - K_t)$$

and computing the limit of the ratio  $\Delta_h(K_t)/h$  for  $h \rightarrow 0$  one obtains

$$\dot{K}(t) = f(K(t)) - K(t)$$

By doing so, the continuous-time version has the same dimensionality as its discrete counterpart. However the two economies, as described by the two versions of the model, are *de facto* very different: capital becomes productive immediately in continuous-time while it took one period (for example one quarter) in discrete-time. This difference emerges as a consequence of the way the time-to-build is measured. In the above explained procedure it coincides with the *period length*, or using a more updated terminology the time-to-build parameter is measured in terms of *frequency of decision making* (see Anagnostopoulos and Giannitsaru [1]).<sup>1</sup> Then the time-to-build structure of capital is lost as the length of the period goes to zero or equivalently the frequency of decision making goes to infinite. On the other hand, there is no economic reason according to which an increase in the frequency of decisions should, all else equals, change the time-to-build structure of capital: rephrasing the well known example of Bohm Bawerk [5], a ship cannot be built in a day even if the agents take their decisions continuously.<sup>2</sup>

In this paper we try to overcome this question, and at the same time to clarify the dimensionality issue, by proposing an alternative way of measuring the time-to-build parameter in terms of period length units.<sup>3</sup> This is done in Section 2, while we study, in Section 3, how this new measure may lead to differences in the qualitative and quantitative dynamic behavior of the economy when we compare the continuous with the discrete-time case. The main qualitative difference emerges when the time-to-build is equal to one period length; in fact the former shows damping output fluctuations in front of monotonic convergence of the latter. On the other hand the two dynamics are very similar, as soon as the time-to-build parameter is an integer multiple of period length  $h$  by a factor greater than one. Interestingly enough, we show that the discrete time model changes its stability properties when its continuous counterpart has an Hopf

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<sup>1</sup>To some extent, we share with these authors the distinction between calendar time and frequency of decision making, but the "objective" of our research is substantially different. In particular it refers to models with time-to-build which are explicitly excluded by their analysis.

<sup>2</sup>See also Licandro and Puch [19] on this point.

<sup>3</sup>A similar measurement issue was pointed out by Burmeister and Turnovsky [11] in their attempt to find the continuous-time version of a discrete-time model with adaptive expectation.

bifurcation.<sup>4</sup>

Finally, at the end of Section 3, we also show that the amplitude of the damping fluctuations in output can be significantly larger in the continuous-time version with respect to its discrete-time counterpart. This last result is one more step in understanding the importance of the time-to-build assumption as a source of output fluctuations (e.g. Wen [22]).

## 2 Description of the Economy

### 2.1 Calendar time, market period and time-to-build.

Let us consider an infinite-horizon economy where the investment and consumption decisions of a representative agent are taken along a discrete sequence of dates:

$$t, t + h, t + 2h, t + 3h, \dots = \{t + sh\}_{s \in \mathbb{N}}$$

where  $t \in \mathbb{R}$  is fixed initial time and the *period length*,  $h$  represents the minimum time interval separating two different and not simultaneous economic decisions. We assume  $h$  to be an integer sub-multiple of a chosen calendar time unit  $\mathcal{T}$ , that is

$$h = \frac{1}{m} \quad \text{in } \mathcal{T} \text{ units}$$

with  $m \in \mathbb{N}$  an integer number.

Now we introduce time-to-build as a basic feature of the economy, and admit any investment to yield new capital after a fixed time delay, whose length is equal to  $J$  when measured in calendar time units. By assuming the time-to-build delay,  $J$ , to be a multiple of the period length  $h$  we may write:

$$\alpha \cdot h = J$$

with  $\alpha \in \mathbb{N}$  an integer number. Combining the two expressions, we may easily observe that

$$J = \frac{\alpha}{m} \quad \text{in } \mathcal{T} \text{ units}$$

with  $\frac{\alpha}{m} \in \mathbb{Q}$  a rational number. We remark that the integer parameter  $\alpha$ , whose value represents the measure of  $J$  in terms of period units  $h$ , will be particularly relevant later when we will show that the dynamics describing the discrete-time version of our model adds up to a difference equations system of order  $2\alpha$ .

As an example, choose the calendar time unit  $\mathcal{T} = 1$  year, the period length  $h$  spanning two months, and a time-to-build value  $J$  of two and a half years. With reference to the symbols above one easily computes:  $h = \frac{1}{6}\mathcal{T}$ ,  $m = 6$ ,  $J = 15h = 15/6\mathcal{T} = 2.5\mathcal{T}$ .

In this context, the decisions' frequency increases as the period length  $h$  decreases. As  $h$  becomes smaller and smaller and converge to zero, we eventually arrive to a limit representation of the model, where the decisions are taken continuously. We call this limit case the *continuous-time version* of the model. It is also worth noting that, for any given  $J$ , the integer ratio  $\alpha = \frac{J}{h}$  will diverge to infinity, because any investment will yield new capital after an infinite number of periods, whenever their length  $h$  tends to zero.<sup>5</sup>

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<sup>4</sup>The proposed procedure to move correctly from discrete to continuous time can be easily extended to vintage capital models (e.g. Boucekkine et al. [6], and [7]).

<sup>5</sup>Regarding this convergence-divergence procedure we stress that it will not depend on the specific form given to the converging sequence  $\{h_k\}$ .

## 2.2 From discrete to continuous-time

The capital accumulation equation of a neoclassical growth model with time-to-build and with a constant return to scale Cobb-Douglas production function, is

$$K_{t+J+sh} = A_h K_{t+sh}^a L_{t+sh}^{1-a} + (1 - \delta_h) K_{t+J+sh-h} - C_{t+sh} \quad (1)$$

where the depreciation rate of capital has been re-scaled as well as the technology parameter, the latter to capture the production  $Y_{t+sh}$  within the interval  $(t + sh, t + (s + 1)h)$ . The dimension of the equation (1) is  $mJ$  which corresponds to the number of economic decisions which can be taken between  $t$  and  $t + J$ .

To complete the description of the economy we also assumed as in Hansen [15] an instantaneous utility function in consumption and labor which allows for indivisible labor supply. After re-scaling the inter-temporal preference discount factor, the social planner problem for this economy can be written as

$$\max \sum_{s=0}^{\infty} \beta_h^s [\log C_{t+sh} - B L_{t+sh}] \quad (2)$$

subject to the capital accumulation equation (1), given the (constant) initial conditions of capital

$$K_i = \bar{K}, \quad \text{with} \quad i = t, t + h, t + 2h, \dots, t + \alpha h = t + J$$

From the Lagrangian, the first order conditions with respect to  $C_{t+sh}$ ,  $L_{t+sh}$ , and  $K_{t+sh+J}$  are

$$\beta_h^s \frac{1}{C_{t+sh}} = \mu_{t+sh} \quad (3)$$

$$\beta_h^s B = \mu_{t+sh} (1 - a) A_h K_{t+sh}^a L_{t+sh}^{-a} \quad (4)$$

$$\mu_{t+sh} = \mu_{t+sh+J} a A_h K_{t+sh}^{a-1} L_{t+sh}^{1-a} + \mu_{t+sh+h} (1 - \delta_h) \quad (5)$$

where  $\mu_{t+sh}$  is the Lagrangian multiplier. These first order conditions hold for any  $s \in \mathbb{N}$  and any  $t \in \mathbb{R}^+ \cup \{0\}$ . Moreover the following intra-temporal substitution condition between consumption and leisure and the Euler equation can be derived easily by (3), (4), and (5):

$$B C_{t+sh} = (1 - a) A_h K_{t+sh}^a L_{t+sh}^{-a} \quad (6)$$

$$\frac{C_{t+sh+h}}{C_{t+sh}} = \beta_h \left[ \beta_h^{\frac{J}{h}-1} \frac{C_{t+sh+h}}{C_{t+J+sh}} a A_h K_{t+sh}^{a-1} L_{t+sh}^{1-a} + 1 - \delta_h \right] \quad (7)$$

It is worth noting that the dimension of this discrete-time model is always finite and equal to  $2\alpha$ .

**Definition 1** *An optimal equilibrium path of the economy is any sequence  $\{C_{t+sh}\}_{s=1}^{\infty}, \{L_{t+sh}\}_{s=1}^{\infty}$ , and  $\{K_{t+sh+J}\}_{s=1}^{\infty}$ , which satisfies equations (1), (6), and (7) plus the initial history of capital and the transversality condition  $\lim_{t+sh \rightarrow \infty} \mu_{t+sh} K_{t+sh} = 0$  for any choice of  $s \in \mathbb{N}$ , and any  $t \in \mathbb{R}^+ \cup \{0\}$ .*

Finally, in order to deduce a continuous-time counterpart from the previous discrete-time dynamical model, we study equations (1), (6) and (7) as the parameter  $h$  shrinks to zero.

**Proposition 1** *Assume the flow variables to be constant within a period and a linear re-scaling of the parameters. Then the continuous-time counterpart of equations (1), (6) and (7) is*

$$\dot{K}(t+J) = AK(t)^a L(t)^{1-a} - \delta K(t+J) - C(t) \quad (8)$$

$$BC(t) = (1-a)AK(t)^a L(t)^{-a} \quad (9)$$

$$\frac{\dot{C}(t)}{C(t)} = \left[ \frac{C(t)}{C(t+J)} e^{-\rho J} aAK(t)^{a-1} L(t)^{1-a} - \delta - \rho \right] \quad (10)$$

**Proof.** See Appendix ■

**Remark 1** *The procedure used to move to continuous time is theoretically consistent since equation (10) coincides with the Euler equation we would have obtained by*

$$\max \int_0^\infty [\log C(t) - BL(t)] e^{-\rho t} dt$$

*subject to the capital accumulation equation (8).*

**Remark 2** *The dimension of the continuous time model becomes infinite since*

$$2\alpha = 2\frac{J}{h} \rightarrow +\infty \quad \text{as} \quad h \rightarrow 0$$

Then the dimension of the model in continuous-time is always infinite as also appears clear from equations (8), (9), and (10) which form a system of mixed functional differential equations, while the dimension of the same model in discrete-time is always finite and equal to  $2\alpha$ . This difference in the dimensionality is not puzzling anymore and is coherent with the procedure used to move from discrete to continuous-time.

**Remark 3** *All the results obtained in this section can be easily extended to a more general framework with a neoclassical utility function and a neoclassical production function.*

**Remark 4** *The model setup, formed by expressions (1) and (2), contains all the specifications of time-to-build generally used by the literature under the assumptions of Proposition 1; in fact*

- a) *If  $h = 1$ , the model describes an economy with gestation lags in production as in Kydland and Prescott ;*
- b) *If  $h \rightarrow 0$ , the model describes an economy with gestation lags as defined by Kalecki [17] and also recently used in growth models (e.g. Asea and Zak [2], and Bambi et al.[4]).*

### 3 Transitional Dynamics

The transitional dynamics of the discrete-time model are studied under the assumptions of Proposition 1. Our local analysis starts with a linearization around the unique steady state  $(C^*, K^*)$  with

$$C^* = \frac{A(1-a)}{B} \left( \frac{a\beta_h^{\frac{J}{h}}(1+\rho h)}{\rho + \delta} \right)^{\frac{a}{1-a}} \quad \text{and} \quad K^* = \frac{\left[ a\beta_h^{\frac{J}{h}}(1+\rho h) \right] C^*}{A(\rho + \delta) - \delta \left[ a\beta_h^{\frac{J}{h}}(1+\rho h) \right]}$$

of the system of equations (1), (6), and (7). After some boring algebra, we obtain

$$\frac{\tilde{k}_{t+sh} - \tilde{k}_{t+sh-h}}{h} = \frac{\delta + \rho}{a} \beta_h^{1-\frac{J}{h}} \tilde{k}_{t+sh-J} - \delta \tilde{k}_{t+sh-h} + \left[ \frac{\rho + \delta}{a^2} \beta_h^{-\frac{J}{h}} - \delta \right] \tilde{c}_{t+sh-J} \quad (11)$$

$$\frac{\tilde{c}_{t+sh+h} - \tilde{c}_{t+sh}}{h} = \frac{\rho + \delta}{1 - \delta h} \tilde{c}_{t+sh} - \frac{\rho + \delta}{a(1 - \delta h)} \tilde{c}_{t+sh+J} \quad (12)$$

where  $\tilde{x}$  indicates the deviation of the variable  $x = \log X$  with  $X = C, K$  from its steady state value  $x^*$ . The characteristic equation associated to this system is the product of the characteristic equation of the homogeneous part of (11) and (12):

$$\underbrace{\left[ \frac{\varpi^\alpha - \varpi^{\alpha-1}}{h} - \frac{\delta + \rho}{a} \beta_h^{1-\alpha} + \delta \varpi^{\alpha-1} \right]}_{=\Delta_k^h(\varpi)} \cdot \underbrace{\left[ \frac{\varpi - 1}{h} + \frac{\rho + \delta}{a(1 - \delta h)} \varpi^\alpha - \frac{\rho + \delta}{1 - \delta h} \varpi \right]}_{=\Delta_c^h(\varpi)} = 0 \quad (13)$$

where  $\varpi$  indicates a generic root of the polynomial  $\Delta_k^h(\varpi) \cdot \Delta_c^h(\varpi)$  whose order is  $2\alpha = 2mJ$ .<sup>6</sup> If half of the roots of this polynomial are outside the unit circle then the Blanchard-Kahn [10] condition for a unique equilibrium path is respected. However this condition can be verified only numerically when the polynomial degree is sufficiently high.<sup>7</sup> For example, when  $h = 1$  quarter and the other parameters' values are chosen as in the Kydland and Prescott's quarterly calibration ( $a = 0.36$ ,  $\delta = 0.025$ , and  $\rho = 0.01$  to yield a return on capital of 1.625% when  $J = 4$ ) we find, for an initial 1% deviation of initial capital from the steady state, saddle-path stability with consumption converging monotonically while capital converging by damping fluctuations

On the contrary, the local dynamics of the economy can be *analytically* derived in continuous-time. By linearizing the system (8), (9), and (10) around the steady state we obtain

$$\dot{\tilde{k}}(t) = \frac{\delta + \rho}{a} e^{\rho J} \tilde{k}(t - J) - \delta \tilde{k}(t) - \left[ \frac{\delta + \rho}{a^2} e^{\rho J} - \delta \right] \tilde{c}(t - J) \quad (14)$$

$$\dot{\tilde{c}}(t) = (\delta + \rho) \tilde{c}(t) - \frac{\delta + \rho}{a} \tilde{c}(t + J) \quad (15)$$

whose characteristic equation is

$$\underbrace{\left[ \lambda - \frac{\delta + \rho}{a} e^{\rho J} e^{-\lambda J} + \delta \right]}_{=\Delta_k(\lambda)} \cdot \underbrace{\left[ z + \frac{\rho + \delta}{a} e^{zJ} - (\rho + \delta) \right]}_{=\Delta_c(z)} = 0 \quad (16)$$

where  $\lambda$  and  $z$  indicate a generic root of the transcendental equations  $\Delta_k(\lambda)$  and  $\Delta_c(z)$  respectively.<sup>8</sup> It is worth noting that the system (14), (15) and the characteristic equation (16) are the continuous-time counterpart of the system (11) (12) and its characteristic equation, (13), which might well be alternatively obtained by computing the limit of the latter as  $h \rightarrow 0$ . The change in dimensionality is now clear since the  $2\alpha$ -degree polynomial becomes the transcendental equation, (16), having infinite roots as  $h \rightarrow 0$ .

<sup>6</sup>The characteristic equation (13) was obtained by searching solution of the form  $\tilde{c}_{t+sh} = \tilde{c}_0 \varpi^s$  and  $\tilde{k}_{t+sh} = \varpi^{s+\alpha} \tilde{k}_{-\alpha}$ .

<sup>7</sup>For example the geometrical method proposed by Grandmont & al., [13] for investigating the stability of a dynamical system can be applied in our case only when  $mJ \leq 1$ .

<sup>8</sup>The distinction between  $\lambda$  and  $z$  is not essential but it will become useful in the proof of Proposition 2 in order to find the equilibrium path of consumption and capital.

From an application of the D-Subdivision method (see for example Bambi [3]) we can learn a lot of information on the spectrum of eigenvalues of  $\Delta_k(\lambda) = 0$  and  $\Delta_c(z) = 0$ . In particular under the assumption that  $J \in (0, \bar{J})$ , with  $\bar{J}$  the unique real root of  $J = \frac{3a\pi}{2(\delta+\rho)}e^{-\rho J}$ , the former has one positive real root,  $\lambda_{\hat{v}}$ , and infinite many complex conjugate roots,  $\lambda_v = x_v + iy_v$ , with negative real part, while the latter has one negative real root,  $\hat{z}$ , and infinite many complex conjugate roots with positive real part. Taking into account these results the following proposition can be proved:

**Proposition 2** *The equilibrium of the economy is locally determinate when  $J \in (0, \bar{J})$ . Moreover the optimal path of consumption converges monotonically while the optimal path of capital by damping fluctuations. Their functional forms are respectively*

$$\tilde{c}(t) = c_0 e^{\hat{z}t} \quad \text{with } c_0 = (\hat{z} - \lambda_{\hat{v}})h'(\lambda_{\hat{v}})n_{\hat{v}} \quad (17)$$

$$\tilde{k}(t) = \left( n_{\hat{v}} - 2 \sum_{v=1}^{+\infty} c_0 \Phi_{1,v} \right) e^{\hat{z}t} + 2 \sum_{v=1}^{+\infty} [(\varsigma_v + c_0 \Phi_{1,v}) \cos y_v t - (\omega_v + c_0 \Phi_{2,v}) \sin y_v t] e^{x_v t} \quad (18)$$

where  $n_v = \varsigma_v + i\omega_v$  indicate the eigenvector associated to the eigenvalue  $\lambda_v$  with  $v \neq \hat{v}$ , while  $\Phi_{1,v}$  and  $\Phi_{2,v}$  are constants. Moreover persistent endogenous fluctuations raise when  $J = \bar{J}$  through an Hopf-bifurcation.<sup>9</sup>

**Proof.** See Appendix ■

Comparing these results with those in the discrete framework with  $m = 1$ , we have also verified numerically that the change in stability in discrete-time arrives at  $J^0$  with  $J^0 = ]\bar{J}[$  where  $]x[$  indicates the smallest integer not less than or equal to  $x$ .<sup>10</sup> It is also worth noting that for  $J = 2$  the discrete-time model has always damping “improper oscillations” in capital because of the presence of two negative and lower than one real roots,<sup>11</sup> and for  $J = 1$  monotonic convergence. This discrepancy with the continuous-time case is due to the fact that the fluctuations arrive at a frequency lower than 1 and then cannot be captured by the discrete-time model when  $m = 1$ . More generally, monotonic convergence in discrete-time emerges as soon as  $J = h$ . These results on stability have been verified for different parameterizations and summarized in the next table.

Finally we have studied if a reduction of the number of decisions,  $m$ , within a calendar-time unit, assumed to be a quarter, may positively affect the amplitude of the output fluctuations under the same calibration of the parameters and the initial condition of capital as before. To this purpose, we have first calculated the output deviations from its steady state value for different choices of  $m$  at some specific calendar dates  $x$ . More precisely we have computed  $\tilde{y}_m(x)$  with  $m = 1, 20, 100, 300$  and  $x = J, J+1, J+2, J+3, 2J$  with  $J = 4$  quarters. The last choice has been made to focus on the time interval where the highest peak takes place. Then we have found the percentage deviations  $D(m, x) = \frac{\tilde{y}_m(x) - \tilde{y}_1(x)}{\tilde{y}_1(x)} \cdot 100$  for the values of  $m$  and  $x$  previously indicated and reported them in Figure 1. What emerges is that increasing the number of operations within a calendar time unit leads to a higher output volatility. More precisely the amplitude percentage difference may be up to 6% higher when we compare the discrete-time

<sup>9</sup>The Hopf bifurcation emerges because at  $J = \bar{J}$  the real part of two conjugate complex roots changes its sign.

<sup>10</sup>See also Bosi and Ragot [8] on the preservation of stability properties and bifurcations in the case of discretization of continuous-time models.

<sup>11</sup>The term “improper oscillations” is also used in Gandolfo [14], page 18, to distinguish this type of oscillations from the trigonometric oscillations.



	discrete-time (period length $h$ )	continuous-time ( $J \neq h \rightarrow 0$ )	continuous-time ( $J = h \rightarrow 0$ )
<b>Local Stability</b>			
No Output Fluctuations	$J = h$	never	always
Damping Output Fluctuations	$h < J < J^0$	$0 < J < \bar{J}$	never
Persistent Output Fluctuations <sup>c</sup>	$J = J^0 = ]\bar{J}[$	$J = \bar{J}$	never
<b>Local Instability</b>			
Explosive Output Fluctuations	$J > J^0 = h$	$J > \bar{J}$	never

<sup>c</sup> The condition indicate the change from local stability to local instability which in continuous-time arrives through an Hopf bifurcation.

model with  $m = 1$  to the continuous-time model with  $m = \infty$ .<sup>12</sup> Intuitively, this result emerges because the agents can increasingly smooth their consumption path as the number of decisions within a calendar-time unit increases and by doing so the output will result more and more volatile.

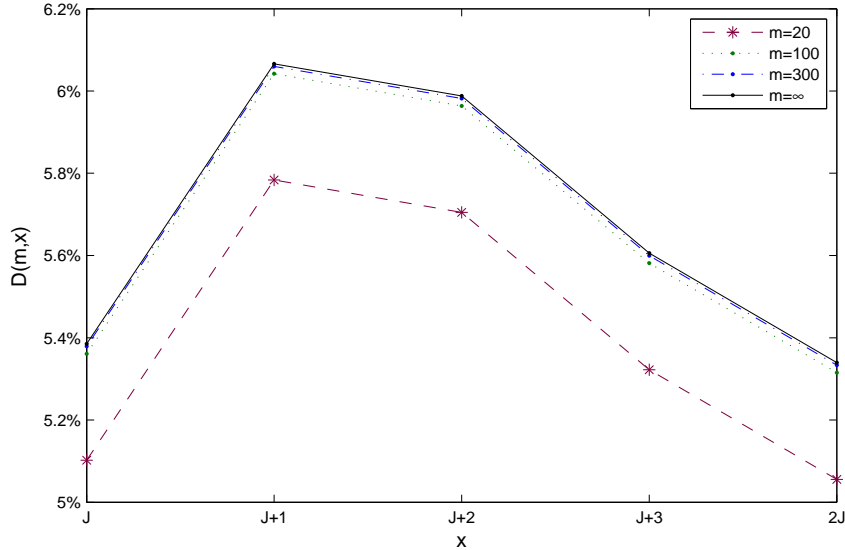


Figure 1: Number of economic decisions,  $m$ , per unit of calendar time and its effect at different dates,  $x$ , on the output volatility,  $D(m, x)$ .

<sup>12</sup>The continuous-time capital path (18) is approximated by computing the first 2000 roots around the imaginary axis using the Lambert function (see for more details Jarlebring and Damm [16]).

## Appendix

**Proof of Proposition 1.** Firstly, the flow variable consumption is re-scaled by assuming a constant rate of change through each unit of period length  $h$ :

$$C_{t+sh} = \int_{t+sh}^{t+(s+1)h} C(t+sh, v) dv = hC(t+sh)$$

Following the terminology used in Turnovsky [21],  $C(t+sh, v)$  indicates the planned (finite) consumption rate for time  $v$ , chosen at time  $t+sh$ , while  $C_{t+sh}$  indicates the cumulated consumption of the representative agent over the interval  $[t+sh, t+(s+1)h]$ .

Secondly, the instantaneous rate of production is assumed to be constant over any calendar time unitary interval. Hence the technological coefficients  $A_h$  and  $A$  - which appear into the functions measuring, respectively, the economy cumulated output along any period length  $h$  and the economy cumulated output along a calendar unitary interval - will be related by the simple linear condition  $A_h = hA$ .

Finally a *linear* re-scaling of the other parameters leads to  $\delta_h = h\delta$ , and  $\beta_h = \frac{1}{1+\rho h}$ .<sup>13</sup>

Then equations (1), (6) and (7) can be rewritten taking into account these two assumptions and their continuous counterpart, namely (8), (9), and (10), can be easily obtained as  $h$  shrinks to zero. ■

### Proof of Proposition 2.

The spectrum of roots of the system is given by the union of the spectrum of roots of  $h_k(\lambda)$ , and  $h_c(z)$ . Since the system is recursively solvable, it is possible to write any solution of (15) using the finite Laplace transform method developed in Belmann and Cooke ([9], pp 197-205):

$$\tilde{c}(t) = \lim_{\ell \rightarrow \infty} \sum_{z_m \in C_\ell} a_m e^{z_m t} \quad (19)$$

where the contour  $C_\ell$ , with  $\ell = 1, 2, \dots$ , cuts the negative and positive imaginary axis to the points  $-y_\ell$ , and  $y_\ell$ , respectively. Moreover  $a_m$  is the residue of the root  $z_m$  and depends on the boundary (initial) condition of consumption. Then the general convergent solution of consumption can be obtained by ruling out all the roots with positive real part, imposing  $a_m = 0 \forall m \neq \hat{m}$ , and then rewriting (19) as

$$\tilde{c}(t) = a_{\hat{m}} e^{\hat{z}t} \quad (20)$$

with  $\hat{z}$  the negative real root of  $h_c(z)$ . On the other hand, the spectrum of  $h_k(\lambda)$  is characterized by one positive real root and all the other complex conjugates and with negative real part when  $J \in (0, \bar{J})$ . The general solution of capital (Belmann and Cooke [9], and Bambi [3]) is

$$\tilde{k}(t) = \sum_v n_v e^{\lambda_v t} - \int_0^t c(s) \sum_v \frac{e^{\lambda_v(t-s)}}{h'_{\lambda_v}} ds \quad (21)$$

$$= \sum_v n_v e^{\lambda_v t} - \int_0^t a_{\hat{m}} e^{\hat{z}s} \sum_v \frac{e^{\lambda_v(t-s)}}{h'_{\lambda_v}} ds \quad (22)$$

$$= \sum_v (n_v + P_v) e^{\lambda_v t} - \sum_v P_v e^{\hat{z}t} \quad (23)$$

<sup>13</sup>It is worth noting that the parameters  $\delta$ ,  $\rho$ ,  $\beta$  and their period length correspondents  $\delta_h$ ,  $\rho_h$ ,  $\beta_h$  are related by the conditions:  $\delta_h = 1 - (1 - \delta)^h$ ,  $\rho_h = (1 + \rho)^h - 1$ , and  $\beta_h = \beta^h$ . Hence,  $h\delta$  and  $h\rho$  are, respectively, the first order approximations of the functions  $\delta_h(\delta)$  and  $\rho_h(\rho)$  in a neighborhood of zero, for any given  $h$ .

where  $n_v$  is the residue of the root  $\lambda_v$ , and  $P_v = \frac{a_{\hat{m}}}{(\lambda_v - \hat{z})h'(\lambda_v)}$ . Moreover capital convergence requires that  $a_{\hat{m}} = c_0$  with

$$c_0 = (\hat{z} - \lambda_{\hat{v}})h'(\lambda_{\hat{v}})n_{\hat{v}} \quad (24)$$

which is the unique condition to rule out the divergent solution induced by  $\lambda_{\hat{v}}$  and to make the transversality conditions hold. Then the optimal path of consumption is monotonic.

On the other hand, (18) can be obtained applying several properties of complex numbers on (23). After imposing  $a_{\hat{m}} = c_0$ , the solution of capital writes

$$\tilde{k}(t) = \sum_{v \neq \hat{v}}^{\infty} (n_v + P_v)e^{\lambda_v t} - \sum_v^{\infty} P_v e^{\hat{z}t} \quad (25)$$

where as shown in Bellman and Cooke [9] the residues are

$$n_v = \frac{\bar{k} + \bar{k}(\lambda_v + \delta) \int_{-J}^0 e^{-\lambda_v s} ds}{1 + J\Lambda e^{-\lambda_v J}} = \frac{\bar{k} \left[ 1 + (\lambda_v + \delta) \left( -\frac{1}{\lambda_v} \right) (1 - e^{\lambda_v J}) \right]}{1 + J\Lambda e^{-\lambda_v J}} \quad (26)$$

with  $\tilde{k}(t) = \bar{k} \in \mathbb{R}^+$  in  $t \in [-J, 0]$ , and  $\Lambda = \frac{\delta + \rho}{a} e^{\rho J}$ .

Now let us call  $\varsigma_v = \text{Re}(n_v)$ ,  $\omega_v = \text{Im}(n_v)$ ,  $\lambda = x + iy$  and  $\bar{\lambda}$  its conjugate, we rewrite

$$\sum_v P_v = -n_{\hat{v}} + c_0 \sum_{v \neq \hat{v}}^{\infty} \left( \frac{1}{(\tilde{z} - \lambda_v)h'(\lambda_v)} + \frac{1}{(\tilde{z} - \bar{\lambda}_v)h'(\bar{\lambda}_v)} \right) \quad (27)$$

which, after some algebra and taking into account the relation  $\bar{\lambda}e^{-\bar{\lambda}J} + \lambda e^{-\lambda J} = 2[x \cos yJ + y \sin yJ]e^{-xJ}$ , becomes

$$\sum_v P_v = -n_{\hat{v}} + 2c_0 \sum_{v \neq \hat{v}}^{\infty} \Phi_{1,v} \quad (28)$$

with

$$\Phi_{1,v} = \frac{(\hat{z} - x_v)(1 + J\Gamma e^{-x_v J} \cos y_v J) - J\Gamma y_v e^{-x_v J} \sin y_v J}{(\hat{z}^2 - 2\tilde{z}x_v + x_v^2 + y_v^2)(1 + J^2\Gamma^2 e^{-2Jx_v} + 2J\Gamma e^{-x_v J} \cos y_v J)} \quad (29)$$

Now let's study the second piece of the solution of capital, namely

$$\sum_{v \neq \hat{v}} (n_v + P_v)e^{\lambda_v t} = \sum_{v \neq \hat{v}} \left( n_v e^{\lambda_v t} + \bar{n}_v e^{\bar{\lambda}_v t} \right) + \sum_{v \neq \hat{v}} \left( P_v e^{\lambda_v t} + \bar{P}_v e^{\bar{\lambda}_v t} \right) \quad (30)$$

$$= 2 \sum_{v \neq \hat{v}} [\varsigma_v \cos y_v t + \omega_v \sin y_v t] e^{-x_v t} + \sum_{v \neq \hat{v}} \left( P_v e^{\lambda_v t} + \bar{P}_v e^{\bar{\lambda}_v t} \right) \quad (31)$$

which after some algebra and taking into account the trigonometric relations  $\cos(a + b) = \cos a \cos b - \sin a \sin b$  and  $\sin(a + b) = \sin a \cos b + \cos a \sin b$  becomes

$$\sum_{v \neq \hat{v}} (n_v + P_v)e^{\lambda_v t} = 2 \sum_{v \neq \hat{v}} [(\varsigma_v + c_0 \Phi_{1,v}) \cos y_v t - (\omega_v + c_0 \Phi_{2,v}) \sin y_v t] e^{x_v t} \quad (32)$$

where

$$\Phi_{2,v} = \frac{y_v + J\Gamma e^{-x_v J} [(\hat{z} - x_v) \sin y_v J + y_v \cos y_v J]}{(\hat{z}^2 - 2\tilde{z}x_v + x_v^2 + y_v^2)(1 + J^2\Gamma^2 e^{-2Jx_v} + 2J\Gamma e^{-x_v J} \cos y_v J)} \quad (33)$$

■

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